

# Target Field Approach for Spherical Coordinates

Ye Bai, Qiuliang Wang, Yunjia Yu, and Keeman Kim

**Abstract**—Using the method of separation of variables and expanding the magnetic field generated by currents flowing on a sphere by means of Fourier-Legendre expansion, we can obtain the inverse relationship between the current density and magnetic flux density. The current density required to generate a specified target field inside the sphere may be evaluated. This approach develops Turner’s target field method and can generate axially directed field in large usable volumes inside the sphere, which can satisfy the requirement of high-energy physics, medical treatment, spatial detection etc. Using this method we can generate uniform or gradient magnetic field that vary linearly along the  $z$  or  $y$  direction. The commercial finite element method software also verifies the validation of this approach.

**Index Terms**—Gradient field, spherical coils, stream function, target field approach.

## I. INTRODUCTION

**I**N FUSION research, high-energy physics, particle accelerators, space detectors and medical imaging devices [1], we often need to generate specific magnetic field in spatial positions according to its application. This type of problem belongs to an inverse method that leads to design the shape of coils knowing the magnetic field distribution. The design of gradient coil is the most common problem. There are two methods of designing gradient coils. One is to adjust the coil position by means of optimum methods step by step until an acceptable design is obtained. Although it is easy to apply for an arbitrary shape of coils, this method has the disadvantage to require heavy mathematical calculations and an exhaustive research of global optimum result.

The other method is called the target field approach [2] that was originally discussed by Turner. It is a noniterative technique which sets predetermined values of magnetic flux density at given spatial positions, and then uses the inverse calculating method to calculate a find back desired current density distribution. Finally by applying the stream function technique to the continuous current distribution, we can generate the discrete current pattern. This approach is mathematically based on Fourier-Bessel expansion of the magnetic field generated by current, thus the relationship between current and field may be inverted. The most typical gradient coil geometry is a cylindrical one where the target field method was used at first. The disadvantages of cylindrically shaped gradient coils are high levels of the dissipated magnetic energies leading to higher values of inductances. Since the stored magnetic energy is proportional to

the fifth power of the coil radius, more compact geometries are preferable, such as the biplanar, elliptical, and spherical coils. In some applications we need to generate specific magnetic field over a spherical volume such as in high-energy physics and MRI [3]. The ideal situation is the spherical coil geometry. In comparison with the cylindrical coils, the spherical coils have the advantage of smaller values of inductances, lower energy storage and less magnetic flux leakage.

In Turner’s article, the target field method is used in cylindrical or shim coils’ design. The subject of generation magnetic field over a spherical volume was studied only by a few researchers. In this paper, Turner’s approach is developed and used in the design of spherical coils. This approach allows the modeling of particular uniform or gradient magnetic fields inside the sphere.

## II. FORMULAS OF SPHERICAL COORDINATES

In source-free region, the magnetostatic field  $B$  can be represented as:

$$B = -\mu_0 \nabla \phi_m.$$

In spherical coordinates, Laplace equation  $\nabla^2 \phi_m = 0$ , can be written as:

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi_m}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \phi_m}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi_m}{\partial \varphi^2} = 0. \quad (1)$$

Using the method of separation variables, we find the expression for  $\phi_m$  to be [4]:

$$\phi_m = \sum_{n=0}^{\infty} \sum_{m=-n}^n A_{mn} r^n P_n^m(\cos \theta) e^{im\varphi} + B_{mn} \frac{1}{r^{n+1}} P_n^m(\cos \theta) e^{im\varphi}. \quad (2)$$

To find the magnetic field everywhere, we first note that the field outside the sphere must decay as  $r \rightarrow \infty$  and the field inside the sphere must be finite as  $r \rightarrow 0$ . Therefore the field outside the sphere can be represented by:

$$B_1 = e_r \left[ \mu_0 \sum_{n=0}^{\infty} \sum_{m=-n}^n B_{mn} \frac{n+1}{r^{n+2}} P_n^m(\cos \theta) e^{im\varphi} \right] - e_\theta \left[ \mu_0 \sum_{n=0}^{\infty} \sum_{m=-n}^n B_{mn} \frac{1}{r^{n+2}} \frac{dP_n^m(\cos \theta)}{d\theta} e^{im\varphi} \right] - e_\varphi \left[ \mu_0 \sum_{n=0}^{\infty} \sum_{m=-n}^n B_{mn} (im) \frac{1}{\sin \theta} \times \frac{1}{r^{n+2}} P_n^m(\cos \theta) e^{im\varphi} \right], \quad (3)$$

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and the field inside the sphere can be represented by:

$$B_Z = -e_r \left[ \mu_0 \sum_{n=0}^{\infty} \sum_{m=-n}^n n A_{mn} r^{n-1} P_n^m(\cos \theta) e^{im\varphi} \right] \\ - e_\theta \left[ \mu_0 \sum_{n=0}^{\infty} \sum_{m=-n}^n A_{mn} r^{n-1} \frac{dP_n^m(\cos \theta)}{d\theta} e^{im\varphi} \right] \\ - e_\varphi \left[ \mu_0 \sum_{n=0}^{\infty} \sum_{m=-n}^n A_{mn} (im) \frac{1}{\sin \theta} \right. \\ \left. \times r^{n-1} P_n^m(\cos \theta) e^{im\varphi} \right]. \quad (4)$$

As we can see, the three respective directions component values are:  $B_r$ ,  $B_\theta$ , and  $B_\varphi$ .

The boundary conditions across the spherical surface are as follows:

$$\begin{cases} n \cdot (B_1 - B_2) = 0 \\ n \times (B_1 - B_2) = \mu_0 J_S. \end{cases} \quad (5)$$

After vector calculation, (5) can be written as[5]:

$$B_{1r} = B_{2r} \quad (6)$$

$$B_{1\theta} - B_{2\theta} = \mu_0 J_\varphi \quad (7)$$

$$B_{1\varphi} - B_{2\varphi} = -\mu_0 J_\theta. \quad (8)$$

Substituting (3), (4) into (6), (7), (8) and multiplying both sides of (7) and (8) by  $e^{-im\varphi}$  and then integrating it from 0 to  $2\pi$  with respect to  $\varphi$  [2], we get:

$$-\sum_{n=0}^{\infty} \frac{2n+1}{n+1} A_{mn} r_0^{n-1} \frac{dP_n^m(\cos \theta)}{d\theta} \\ = -\frac{1}{2\pi} \int_0^{2\pi} e^{-im\varphi} J_\varphi d\varphi. \quad (9)$$

$$\sum_{n=0}^{\infty} \frac{2n+1}{n+1} A_{mn} (im) \frac{1}{\sin \theta} r_0^{n-1} P_n^m(\cos \theta) \\ = -\frac{1}{2\pi} \int_0^{2\pi} e^{-im\varphi} J_\theta d\varphi. \quad (10)$$

The associated Legendre functions are subject to the following orthogonal relationship [6]:

$$\int_0^\pi \left[ \frac{\partial P_n^m(\cos \theta)}{\partial \theta} \frac{\partial P_{n1}^m(\cos \theta)}{\partial \theta} \right. \\ \left. + \frac{m^2}{\sin^2 \theta} P_n^m(\cos \theta) P_{n1}^m(\cos \theta) \right] \sin \theta d\theta \\ = \delta_{nn1} \frac{2n(n+1)(n+m)!}{(2n+1)(n-m)!} \\ \delta_{nn1} = \begin{cases} 1 & n = n1 \\ 0 & \text{else.} \end{cases} \quad (11)$$

Using (9), (10) and (11) we finally obtain [7]:

$$A_{mn} = \frac{1}{r_0^{n-1}} \frac{(n-m)!}{2n(n+m)!} \left( j_\varphi^{(m)} + j_\theta^{(m)} \right), \quad (12)$$

where

$$j_\varphi^{(m)} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^\pi \frac{dP_n^m(\cos \theta)}{d\theta} \sin \theta J_\varphi e^{-im\varphi} d\varphi d\theta \quad (13)$$

$$j_\theta^{(m)} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^\pi (im) P_n^m(\cos \theta) J_\theta e^{-im\varphi} d\varphi d\theta. \quad (14)$$

The value of z component of magnetic flux density inside the sphere can be written as:

$$B_Z = B_r \cos \theta - B_\theta \sin \theta. \quad (15)$$

The associated Legendre functions are subject to the following recurrent relationship [6]:

$$\frac{dP_n^m(\cos \theta)}{d\theta} = -\frac{1}{\sin \theta} \left[ (n+m) P_{n-1}^m(\cos \theta) \right. \\ \left. - n \cos \theta P_n^m(\cos \theta) \right]. \quad (16)$$

Substituting (4), (16) into (15), multiplying both sides of (15) by  $e^{-im\varphi}$  and then integrating it from 0 to  $2\pi$  with respect to  $\varphi$ , we get:

$$\frac{1}{2\pi} \int_0^{2\pi} B_Z e^{-im\varphi} d\varphi \\ = -\mu_0 \sum_{n=0}^{\infty} A_{m(n+1)} r^n (n+m+1) P_n^m(\cos \theta). \quad (17)$$

The associated Legendre functions are also subject to the following orthogonal relationship [6]:

$$\int_0^\pi P_n^m(\cos \theta) P_{n1}^m(\cos \theta) \sin \theta d\theta = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} \delta_{nn1}. \quad (18)$$

Considering the orthogonal relationship (18), we substitute (12) into (17), multiply both sides of (17) by  $P_n^m(\cos \theta) \sin \theta$  and then integrate it from 0 to  $\pi$  with respect to  $\theta$ . After simplification, we finally obtain an important relationship:

$$b_Z^{(m)} = -\mu_0 \frac{(n+1-m)}{(n+1)(2n+1)} \left( \frac{r}{r_0} \right)^n \left( j_\varphi^{(m)} + j_\theta^{(m)} \right) \quad (19)$$

where

$$b_Z^{(m)} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^\pi P_n^m(\cos \theta) \sin \theta B_Z e^{-im\varphi} d\varphi d\theta. \quad (20)$$

Equation (19) is the general Fourier transform relationship between the axially z-directed field inside the sphere and the spherical surface current density.

The value of y component of magnetic flux density inside the sphere can be written as:

$$B_Y = B_r \sin \theta + B_\theta \cos \theta. \quad (21)$$

By means of a similar derivation method, we can obtain the following relationship:

$$b_Y^{(m)} = \mu_0 \left( \frac{r}{r_0} \right)^n \left[ \frac{(n+1-m)!}{(m+n)!(2n+1)(2n+3)} \right. \\ \left. - \frac{1}{(2n+3)} \right] \left( j_\varphi^{(m)} + j_\theta^{(m)} \right) \quad (22)$$

where

$$b_Y^{(m)} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^\pi P_n^m(\cos \theta) \sin^2 \theta B_Y e^{-im\varphi} d\varphi d\theta \quad (23)$$

and  $j_\varphi^{(m)}$ ,  $j_\theta^{(m)}$  are as defined earlier.

According to the above derivation, the general principles of this approach is first to specify a desired magnetic field inside the spherical volume, then using (19) or (22) to obtain the general Fourier transform of the surface current density. At last, we can perform the general inverse Fourier transform to find the desired surface current values of  $J_\varphi$  and  $J_\theta$  on the sphere.

### III. FORMULAS APPLICATION

We assume the desired distribution of the z component of magnetic flux density to be

$$B_Z = \frac{2}{3}\mu_0 \frac{J_0}{r_0}$$

that means a uniform magnetic flux density.  $J_0$  represents the value of the current density. According to (13), (14), (18), (19) and (20) we can finally obtain:

$$J_\varphi = \frac{J_0}{r_0} \sin \theta, \quad J_\theta = 0.$$

Resorting to the above results and (22), (23), we also obtain

$$B_Y = 0.$$

Another example is to assume the desired gradient field along the z-axis as

$$B_Z = \frac{2\mu_0 J_0 z}{5r_0^2}.$$

Applying the same technique we finally get:

$$J_\varphi = \frac{J_0}{r_0} \sin \theta \cos \theta, \quad J_\theta = 0, \quad B_Y = -\frac{\mu_0 J_0 y}{5r_0^2}.$$

By means of the above method, we can generate spherical coils that are capable of producing an axially directed field, that is uniform or vary linearly along the z direction or y direction. This conclusion can also be partly verified from [5].

### IV. DISCRETE APPROXIMATION OF THE CONTINUOUS CURRENT DENSITY

Following the target field approach, the outcome is the continuous current distribution for the spherical coils. In order to get the actual axially uniform or gradient coils, we must generate the discrete current distribution patterns to fabricate the coils. To use this method we must know information about the number of the individual current loops in each section, their position relative to the center of the sphere and the magnitude of the current in the current loops. This is feasible by using the stream function technique that was originally presented by W. A. Edelstein [8]. Since the current density satisfies the continuity equation, there exists a vector function  $\mathbf{S}$ , such that

$$J_S = \nabla \times \mathbf{S}.$$

In the case of spherical coils mentioned above, the current density is confined to lie on the surface of the sphere and is directed along the  $\varphi$  direction. Expressing the curl of a vector in spherical coordinates, we obtain:

$$J_\varphi = \frac{1}{r_0} \frac{\partial}{\partial \theta} S_\theta. \quad (24)$$

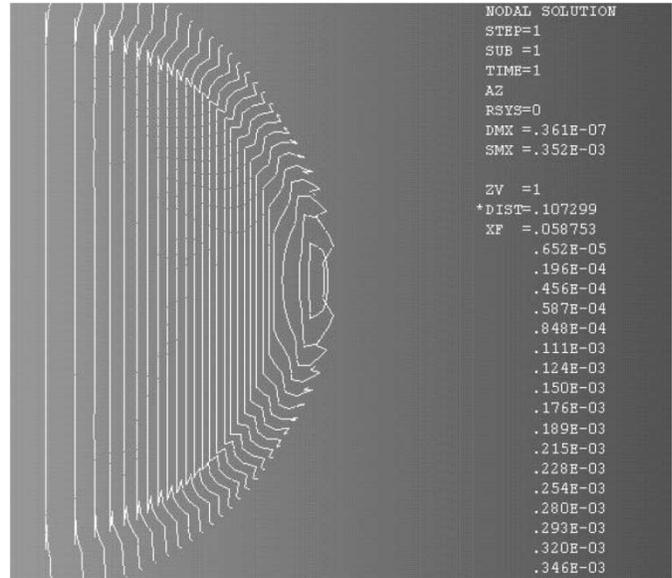


Fig. 1. Distribution of flux lines inside the sphere.

Then by integrating the expression over the line segments, we determine the position component  $S_i$  of the individual loops as:

$$S_i = r_0 \int_{\theta_i}^{\theta_{i+1}} J_\varphi(\theta) d\theta. \quad (25)$$

The number of wires in each region is determined using the relation:

$$N_i = \frac{S_i}{I}, \quad I = \frac{S_{total}}{N_{total}}$$

where

$$S_{total} = r_0 \int_0^\pi |J_\varphi(\theta)| d\theta. \quad (26)$$

$N_{total}$  stands the whole number of the current loops on the sphere surface, which is determined before the coil's construction. Symbol  $I$  means the magnitude of a current loop.

Using this method, we can use discrete coils to substitute the continuous current density on the sphere surface. Although it will bring out some errors, we can diminish them by increasing the current loops.

### V. VERIFICATION OF THE APPROACH BY FINITE ELEMENT METHOD

To verify this approach we used a commercial finite element method analysis software [9] to simulate the spherical coils mentioned above. For succinctness only a semi sphere is shown. Fig. 1 shows the flux lines inside the sphere generated by the current density

$$J_\varphi = \frac{J_0}{r_0} \sin \theta, \quad J_\theta = 0$$

on the surface of the sphere. The z component of flux density inside the sphere is shown in Fig. 2. As we can see, the z-axis uniform magnetic field has been generated. Because the y component of flux density is zero, we do not plot this figure.

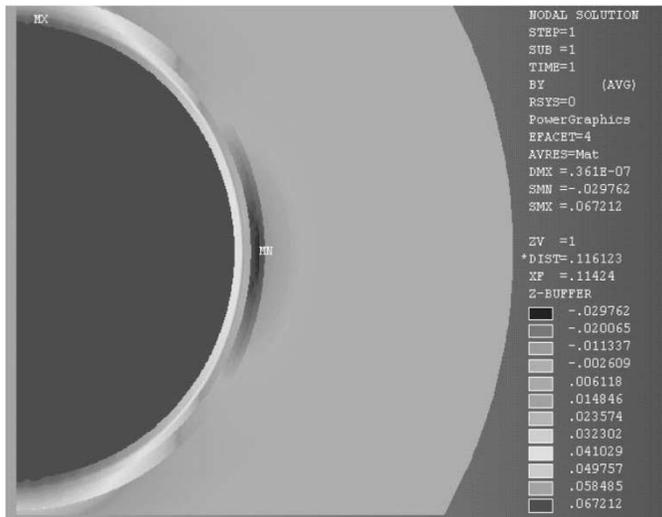
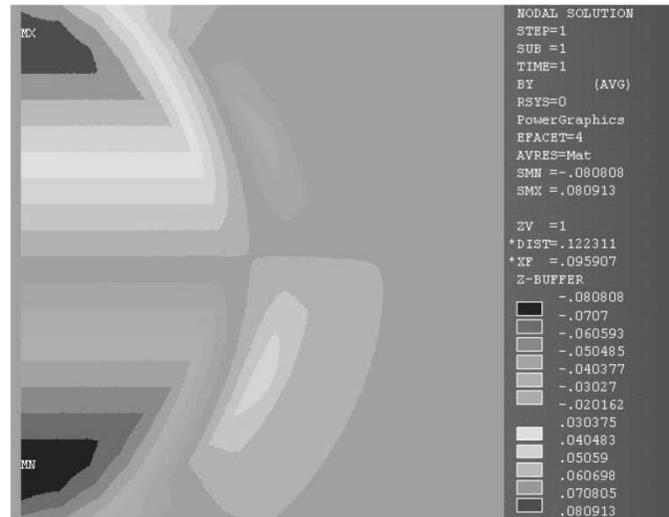
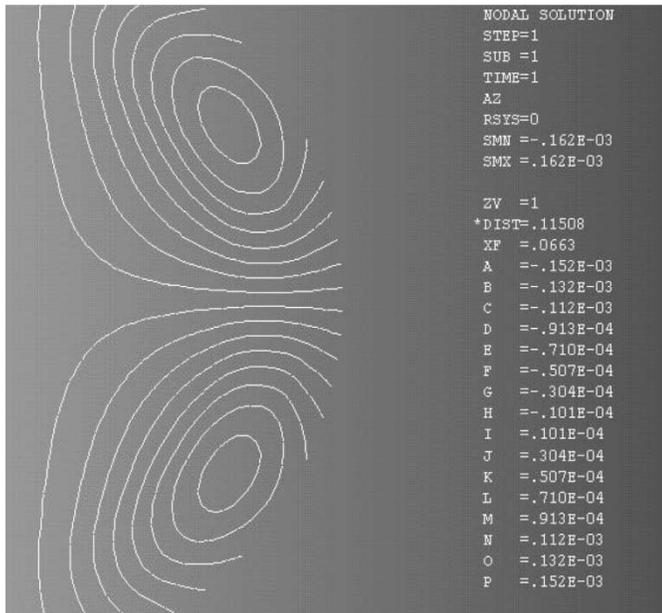
Fig. 2. Z-axis uniform field  $B_z$  inside the sphere.Fig. 4. Z-axis gradient field  $B_z$  inside the sphere.

Fig. 3. Distribution of flux lines inside the sphere.

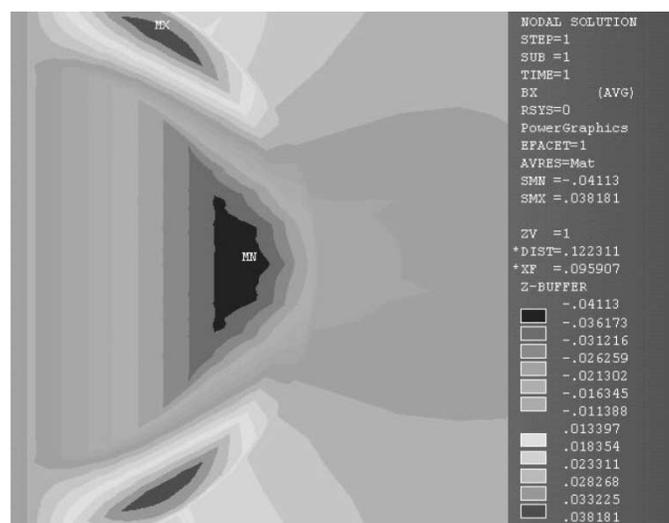
Fig. 3 shows the flux lines inside the sphere generated by the current density

$$J_\varphi = \frac{J_0}{r_0} \sin \theta \cos \theta, \quad J_\theta = 0$$

on the spherical surface. The z and y components of the flux density inside the sphere are shown in Figs. 4 and 5. As clearly shown, we can generate gradient field that vary linearly along the z-axis and y-axis inside the sphere.

## VI. CONCLUSION

The target field approach for spherical coordinates provides a powerful tool in the design of spherical coils. Applying an inverse calculating method, given the z or y component of magnetic flux density inside the sphere we can obtain the surface current density on the sphere. Then by using the stream function technique, we substitute the continuous current density on

Fig. 5. Y-axis gradient field  $B_y$  inside the sphere.

the sphere surface by the means of discrete coils. This technique is appropriate for engineering design.

In particle accelerators, electron beam devices, source for bi-asing fields and today's ultra fast MRI applications, a highly uniform or linearly gradient field is often required in a spherical volume. The method, described in this article, develops Turner's target field approach and can greatly simplify the design process of spherical coils. Finally, we use the commercial finite element method software ANSOFT to verify this approach and obtain satisfactory simulating results.

## NOMENCLATURE

Symbols	Quantity SI
$\phi_m$	Magnetic scalar potential. [A]
$A_{mn}$	Arbitrary constants to be determined by specific problems.
$B_{mn}$	Arbitrary constants to be determined by specific problems.
$P_n^m(\cos \theta)$	Associated Legendre Function.

<b>B</b>	Vector of Magnetic flux density. [T]	$j_{\theta}^{(m)}$	General Fourier transform of $J_{\theta}$
$\mu_0$	Magnetic permeability of free space. [ $4\pi \times 10^{-7}$ H/m]	<b>S</b>	Vector stream function
$\mathbf{e}_r$	Unit vector along the radial direction.	$N_i$	Number of wires in each region
$\mathbf{e}_{\theta}$	Unit vector along the $\theta$ direction.	$S_i$	Value of the current function in each region
$\mathbf{e}_{\varphi}$	Unit vector along the $\varphi$ direction.	$I$	Desired magnitude of the current in the loop [A]
$r_0$	Radius of the sphere [m]		
$B$	Column matrix of flux density components. [T]		
<b>n</b>	Unit vector defining the normal direction of sphere.		
<b>J<sub>S</sub></b>	Vector of current density. [A/m <sup>2</sup> ]		
$J_{\varphi}$	$\varphi$ component of current density. [A/m <sup>2</sup> ]		
$J_{\theta}$	$\theta$ component of current density. [A/m <sup>2</sup> ]		
$\delta_{nm1}$	Dirac's symbol.		
$\nabla$	Hamilton operator.		
$b_Z^{(m)}$	General Fourier transform of $B_Z$ .		
$b_Y^{(m)}$	General Fourier transform of $B_Y$ .		
$j_{\varphi}^{(m)}$	General Fourier transform of $J_{\varphi}$ .		
$(dP_n^m)$	Derivative of associate Legendre polynomial.		
$(\cos \theta)/(d\theta)$			

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